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Non-Linear Diffusion III. Diffusion through Isotropic Highly Elastic Solids

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NON-LINEAR DIFFUSION
III. DIFFUSION THROUGH ISOTROPIC HIGHLY ELASTIC SOLIDS

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CONTENTS

A theory is formulated for the diffusion of fluids through highly elastic materials. This forms an extension of earlier work (Adkins 1963 *a*, *b*) on flows of mixtures of fluids, but attention is here confined to the diffusion of a single fluid through the solid. It is assumed that each point of the region of space concerned may be occupied simultaneously by the fluid and solid, and the motion of each constituent is governed by the usual equations of motion and continuity. The mechanical properties of each substance are specified by means of constitutive equations for the stresses, and diffusion effects by means of a body force or diffusive drag acting on each component, this force being a function of the composition and relative motions of the constituents of the solid-fluid mixture.

In applications, attention is confined to steady-state problems. These include the swelling of a solid due to presence of fluid, and the diffusion of a fluid through uniform plane plates or slabs subject to uniform all-round compression or extension and to a shearing deformation. It is assumed throughout that the diffusing fluid is non-Newtonian, the theory for ideal and viscous fluids emerging as special cases.

1. Introduction

In earlier papers (Adkins 1963 *a*, *b*) a non-linear theory for the diffusion and flows of mixtures of fluids has been formulated, based on an approach suggested by Truesdell & Toupin (1960), who give a general discussion of the problem with extensive references. This approach, which was subsequently examined with special reference to linear theories by Truesdell (1961, 1962) avoids some of the fundamental difficulties of the classical theory based upon Fick's law. Accounts of the classical theory with applications are given by Crank (1956), Bird, Stewart & Lightfoot (1960) and others.

In the present work, the non-linear theory is extended to deal with the diffusion of fluids through highly elastic, isotropic solids. This problem arises in many important applications such as the swelling of rubber by solvents, the absorption of oils by plastics and of water by fibres and the seepage of water and other fluids through porous media. The latter problem has been examined by Biot (1956*a*, *b*).

The assumptions are similar to those of the earlier work. Each point of the solid-fluid mixture is assumed to be occupied simultaneously by all constituents in given proportions. Mechanical and kinematic quantities such as density, a stress tensor and body-force vector, and velocity and acceleration vectors are defined for each constituent, and equations of motion and of continuity are formulated for each substance using these quantities.

To simplify the theory attention is confined to the situation where a single non-Newtonian fluid is diffusing through an ideally elastic solid. An extension to deal with a mixture of fluids and with more general materials presents no difficulty of principle. During the diffusion process it is assumed that the stresses for a given substance describe its internal properties. The stress components for the solid $\mathscr{S}_1$ then depend upon the density and the deformation gradients defined for $\mathscr{S}_1$; the stresses for the fluid $\mathscr{S}_2$ depend upon the density and the velocity gradients defined for $\mathscr{S}_2$. More general assumptions in which terms are included describing interactions between materials have been made in some of the earlier work (Adkins 1963*b*), and a brief indication of some of the generalizations possible to include interactions is given here.

The effect of diffusion is taken into account by means of a body force acting on each constituent. This diffusive force depends upon the nature of the mixture and the relative motion of its constituents. In the present instance we suppose that the diffusive force acting on the fluid (which is balanced by that acting on the solid) depends upon the densities of solid and fluid, the velocity of the fluid relative to the solid, and the state of deformation of the solid. Since velocity gradients occur in the constitutive equation for the fluid we assume that these may also affect the diffusive force. The constitutive equations are restricted by invariance considerations.

The theory is applied to examine some steady-state problems, including the swelling of a solid by absorption of fluid, diffusion through a uniformly stretched plate and diffusion through a sheared slab. In these latter cases the deformation becomes non-uniform as a result of the diffusion process and when diffusion takes place through a sheared solid without a centre of symmetry a shearing deformation and flow may be produced normal to the plane of the initial shear.

2. Notation and formulae

We consider a highly elastic solid $\mathscr{S}_1$ which is undergoing a continuous deformation, and suppose that the region $\mathscr{R}$ occupied by the solid is also permeated by a fluid $\mathscr{S}_2$ which is in motion relative to $\mathscr{S}_1$. Each point of $\mathscr{R}$ is therefore occupied simultaneously by the solid $\mathscr{S}_1$ and the fluid $\mathscr{S}_2$, these proportions varying, in general with time and with position in $\mathscr{R}$.

The motion is referred to a fixed system of rectangular Cartesian co-ordinates X_i. At some initial time $t = 0$, we suppose that the solid and fluid are both at rest, that the solid $\mathscr{S}_1$ is undeformed and that the region occupied by it does not contain any of the fluid $\mathscr{S}_2$. At subsequent time $t\,(>0)$ the fluid has diffused into the solid, which is deformed by its action and by extraneous body and surface forces. Particles of the substances $\mathscr{S}_1$, $\mathscr{S}_2$ which are at the point y_i at the current time t were at X_i, Y_i, respectively, at the initial time

$t = 0$. The velocities of $\mathscr{S}_1$, $\mathscr{S}_2$ at y_i at time t are $\mathbf{u}$, $\mathbf{v}$ with components u_i, v_i, respectively. The motions of $\mathscr{S}_1$, $\mathscr{S}_2$ are therefore defined by

$$\left.\begin{aligned} y_i &= y_i(X_k, t), \quad u_i = \frac{^{(1)}\mathrm{D}y_i}{\mathrm{D}t} \quad (\text{solid } \mathscr{S}_1), \\ y_i &= y_i(Y_k, t), \quad v_i = \frac{^{(2)}\mathrm{D}y_i}{\mathrm{D}t} \quad (\text{fluid } \mathscr{S}_2), \end{aligned}\right\} \tag{2·1}$$

where $^{(1)}\mathrm{D}/\mathrm{D}t$, $^{(2)}\mathrm{D}/\mathrm{D}t$ denote differentiation with respect to t holding the co-ordinates X_i, Y_i constant, respectively.

The operators $^{(1)}\mathrm{D}/\mathrm{D}t$, $^{(2)}\mathrm{D}/\mathrm{D}t$ are given by

$$\frac{^{(1)}\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u_m \frac{\partial}{\partial y_m}, \quad \frac{^{(2)}\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + v_m \frac{\partial}{\partial y_m}, \tag{2·2}$$

where $\partial/\partial t$ denotes differentiation with respect to t holding the co-ordinates y_i constant and here and subsequently, summation is carried out over repeated indices unless otherwise indicated. When operating on any quantity $\phi = \phi(t)$ which is independent of position

$$^{(1)}\mathrm{D}/\mathrm{D}t = {}^{(2)}\mathrm{D}/\mathrm{D}t = \partial/\partial t.$$

The densities of $\mathscr{S}_1$, $\mathscr{S}_2$ at the point y_i at time t are denoted by ρ_1, ρ_2 respectively; the initial density of the solid $\mathscr{S}_1$ in the unstressed and unstrained state at time $t = 0$ is denoted by ρ_0. In the absence of chemical reactions between $\mathscr{S}_1$, $\mathscr{S}_2$ we have the equations of continuity

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial y_i}(\rho_1 u_i) = 0, \quad \frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial y_i}(\rho_2 v_i) = 0, \tag{2·3}$$

or alternatively, for the solid $\mathscr{S}_1$

$$\sqrt{I_3} = \frac{\rho_0}{\rho_1} = \left|\frac{\partial y_i}{\partial X_k}\right|. \tag{2·4}$$

In (2·4), I_3 is the third strain invariant (see, for example, Green & Adkins 1960); if $\mathscr{S}_1$ remains incompressible throughout the deformation and the diffusion process, $\sqrt{I_3} = 1$.

For each constituent $\mathscr{S}_1$, $\mathscr{S}_2$ of the solid-fluid mixture we postulate that there exists at the point y_i a partial stress tensor $\boldsymbol{\sigma}_\alpha$ with components $\sigma^{(\alpha)}_{ij}$ $(\alpha = 1, 2)$ and extraneous and diffusive body forces $\mathbf{F}_\alpha$, $\boldsymbol{\Psi}_\alpha$ per unit mass with components $F^{(\alpha)}_i$, $\Psi^{(\alpha)}_i$, respectively, referred to the X_i-axes. For each substance $\mathscr{S}_\alpha$ we may then formulate equations of motion

$$\frac{\partial \sigma^{(\alpha)}_{ik}}{\partial y_k} + \rho_\alpha(F^{(\alpha)}_i + \Psi^{(\alpha)}_i) = \rho_\alpha \frac{^{(\alpha)}\mathrm{D}v^{(\alpha)}_i}{\mathrm{D}t} \quad (\alpha = 1, 2;\ v^{(1)}_i = u_i,\ v^{(2)}_i = v_i;\ \alpha \text{ not summed}). \tag{2·5}$$

In subsequent work we shall restrict attention to the case in which extraneous body forces are absent. The diffusive forces $\boldsymbol{\Psi}_\alpha$ arise from interactions between the solid and fluid and we may therefore write

$$\rho_1 \boldsymbol{\Psi}_1 + \rho_2 \boldsymbol{\Psi} = \mathbf{0}, \quad (\boldsymbol{\Psi} = \boldsymbol{\Psi}_2,\ \Psi^{(2)}_i = \Psi_i). \tag{2·6}$$

In these circumstances, the equations (2·5) yield

$$\left.\begin{aligned} \frac{\partial \sigma^{(1)}_{ik}}{\partial y_k} - \rho_2 \Psi_i &= \rho_1\left(\frac{\partial u_i}{\partial t} + u_m \frac{\partial u_i}{\partial y_m}\right), \\ \frac{\partial \sigma^{(2)}_{ik}}{\partial y_k} + \rho_2 \Psi_i &= \rho_2\left(\frac{\partial v_i}{\partial t} + v_m \frac{\partial v_i}{\partial y_m}\right). \end{aligned}\right\} \tag{2·7}$$

3. Stresses and diffusive forces: physical assumptions

The system of equations (2·3), (2·4) and (2·7) is completed by the introduction of constitutive equations for the stresses $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and the diffusive force $\boldsymbol{\Psi}$.

A variety of assumptions is possible, the simplest being that the stresses in a given substance describe internal mechanical properties which are not affected by the presence of other materials, while the force $\boldsymbol{\Psi}$ accounts entirely for the effects of interactions arising from diffusion. Assumptions of this kind, excluding interaction terms in the stresses have been made in an earlier paper (Adkins 1963 *a*) in considering mixtures of fluids. Here, we confine attention to the situation in which $\mathscr{S}_1$ is ideally elastic in the absence of $\mathscr{S}_2$, while $\mathscr{S}_2$, in the absence of $\mathscr{S}_1$ is a non-Newtonian fluid. If interactions are excluded $\boldsymbol{\sigma}_1$ depends upon ρ_1 and the displacement gradients $\partial y_i/\partial X_k$, while $\boldsymbol{\sigma}_2$ depends upon ρ_2 and the velocity gradients $\partial v_i/\partial y_k$. Thus

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1(\rho_1, \partial y_i/\partial X_k), \quad \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_2(\rho_2, \partial v_i/\partial y_k). \tag{3·1}$$

The diffusive force $\boldsymbol{\Psi}$ may be regarded as a retarding effect exerted upon the motion of the fluid $\mathscr{S}_2$ due to the presence of the elastic solid. It is natural to suppose that it depends upon the density of fluid ρ_2, its velocity relative to $\mathscr{S}_1$ and the nature of the medium through which it is diffusing; this is described by the density ρ_1 and the displacement gradients $\partial y_i/\partial X_k$. Since the fluid stresses depend upon velocity gradients it is conceivable that these should also enter into the expression for the diffusive force. These considerations lead to the functional form

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}\left(\rho_1, \rho_2; \frac{\partial y_i}{\partial X_k}; \frac{\partial u_i}{\partial y_k}, \frac{\partial v_i}{\partial y_k}; U_i\right), \tag{3·2}$$

for $\boldsymbol{\Psi}$ where U_i are the components of the relative velocity vector

$$\mathbf{U} = \mathbf{v} - \mathbf{u}. \tag{3·3}$$

To examine invariance requirements for $\boldsymbol{\Psi}$ it is convenient to consider a scalar function

$$\mathscr{H} = p_i \Psi_i = \mathscr{H}\left(p_i; \rho_1, \rho_2; \frac{\partial y_i}{\partial X_k}; \frac{\partial u_i}{\partial y_k}, \frac{\partial v_i}{\partial y_k}; U_i\right), \tag{3·4}$$

which is linear and homogeneous in the components p_i of an arbitrary vector $\mathbf{p}$; (3·4), together with the forms derived from it, is assumed to be single-valued and continuous in ρ_1, ρ_2 and a polynomial in the remaining arguments. When the form of $\mathscr{H}$ has been determined, the diffusive force components are given uniquely by

$$\Psi_i = \partial \mathscr{H}/\partial p_i. \tag{3·5}$$

In the present paper, attention will be concentrated on the implications of the simple theory based upon (3·1) and (3·2). More generally, we may suppose that interaction terms occur in the stresses; a simple hypothesis of this kind is expressed by the relations

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1\left(\rho_1, \rho_2, \frac{\partial y_i}{\partial X_k}\right), \quad \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_2\left(\rho_1, \rho_2, \frac{\partial v_i}{\partial y_k}\right), \tag{3·6}$$

in which the stresses each depend upon both densities ρ_1, ρ_2. Some support for interactions of this type is provided by the results for swelling discussed in § 7. In (3·6) we must have $\boldsymbol{\sigma}_1 = \mathbf{0}$ when $\rho_1 = 0$ and $\boldsymbol{\sigma}_2 = \mathbf{0}$ when $\rho_2 = 0$. A more general formulation in which the stresses take forms comparable with the diffusive force (3·2) is developed elsewhere for aeolotropic bodies (Adkins 1964); a somewhat different approach, in which elastic terms are excluded from $\boldsymbol{\sigma}_2$ and the only additional terms appearing in $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are those which are excluded by invariance considerations from either constituent alone, is given by Green & Adkins (1964). In both cases the analysis of the invariance problem is similar to that given here for the diffusive force.

4. Diffusion through an isotropic elastic solid

The reduction of the expression (3·4) for $\mathscr{H}$ to a form appropriate to an isotropic fluid diffusing through an isotropic elastic solid follows the lines of the earlier work (1963 *b*). Here, however, to simplify the appearance of the expressions derived, we use matrix notation and define mechanical and kinematic matrices by the relations

$$\boldsymbol{\sigma}_\alpha = \|\sigma_{ik}^{(\alpha)}\|, \tag{4·1}$$

$$\left.\begin{aligned} &A = \|A_{ik}\|, \quad B = \|B_{ik}\|, \quad C = \|C_{ik}\|, \\ &A_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial y_k}+\frac{\partial u_k}{\partial y_i}\right), \quad B_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_k}+\frac{\partial v_k}{\partial y_i}\right), \\ &C_{ik} = \frac{\partial y_i}{\partial X_r}\frac{\partial y_k}{\partial X_r}, \end{aligned}\right\} \tag{4·2}$$

$$\left.\begin{aligned} &R_1 = \|R_{ik}^{(1)}\|, \quad R_2 = \|R_{ik}^{(2)}\|, \quad R = \|R_{ik}\|, \\ &R_{ik}^{(1)} = \frac{1}{2}\left(\frac{\partial u_i}{\partial y_k}-\frac{\partial u_k}{\partial y_i}\right), \quad R_{ik}^{(2)} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_k}-\frac{\partial v_k}{\partial y_i}\right), \\ &R_{ik} = R_{ik}^{(1)}-R_{ik}^{(2)}. \end{aligned}\right\} \tag{4·3}$$

Vectors with components X_i, Y_i, y_i are denoted by $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{y}$ respectively and $M = \|M_{ik}\|$ denotes the orthogonal matrix for which

$$MM^T = M^TM = I, \quad \text{(i)} \qquad \det M = 1, \quad \text{(ii)} \tag{4·4}$$

I being the unit matrix and M^T the transpose of M.

If $\mathscr{M}$, $\mathscr{M}_r$ are two motions of the mixture differing only to the extent of an arbitrary rigid body motion, and a given particle P is at $\mathbf{y}$, $\bar{\mathbf{y}}$ respectively at time t in these two motions, then

$$\bar{\mathbf{y}} = M\mathbf{y}, \tag{4·5}$$

where $M = M(t)$ is in general a function of time. A corresponding rigid body rotation of the undeformed solid $\mathscr{S}_1$ is defined by

$$\bar{\mathbf{X}} = M\mathbf{X}, \tag{4·6}$$

M now being a constant matrix.

In the present notation, the arguments in (3·4) may be rearranged so that $\mathscr{H}$ takes the form

$$\mathscr{H} = \mathscr{H}\left(\mathbf{p}; \rho_1, \rho_2; \frac{\partial y_i}{\partial X_k}; A, B, R, \mathbf{U}, R_1\right). \tag{4·7}$$

In (4·7) only the displacement gradients $\partial y_i/\partial X_k$ are affected by the rigid body rotation defined by (4·6) of the undeformed solid $\mathscr{S}_1$ and it follows by the argument used in considering the strain energy function for elastic materials (Green & Adkins 1960) that if $\mathscr{S}_1$ has (hemihedral) isotropy, $\partial y_i/\partial X_k$ can enter into (4·7) only in the combinations C_{ik} and $\surd I_3$, the polynomial character of $\mathscr{H}$ being preserved. Since $\surd I_3 = \rho_0/\rho_1$, we may, without loss of generality exclude this argument provided $\mathscr{H}$ can still be regarded as single valued and continuous in ρ_1.

If we now examine the behaviour of the arguments in $\mathscr{H}$ for the motions $\mathscr{M}$, $\mathscr{M}_r$ and distinguish quantities appropriate to $\mathscr{M}_r$ by bars, we find that $\bar{R}_1$ involves the angular velocity of the superposed rigid body motion contained in $\mathscr{M}_r$ and for the remaining quantities we have

$$\left.\begin{aligned}\bar{\mathbf{Q}} &= M\mathbf{Q} \qquad (\mathbf{Q} = \mathbf{p} \text{ or } \mathbf{U}),\\ \bar{\chi} &= M\chi M^T \quad (\chi = A, B, C \text{ or } R).\end{aligned}\right\} \tag{4·8}$$

The elements of R_1 cannot therefore appear in $\mathscr{H}$, which reduces to the form

$$\mathscr{H} = \mathscr{H}(\mathbf{p};\, \rho_1, \rho_2;\, C, A, B, R, \mathbf{U}). \tag{4·9}$$

Since $\mathscr{H}$ is independent of the rigid body motions specified by (4·4), (4·5), it is a hemihedral isotropic function of the vectors $\mathbf{p}$, $\mathbf{U}$ and the matrices C, A, B, R.*

The invariants of an arbitrary system of vectors and second-order symmetric tensors under transformations of the orthogonal group have been examined by Spencer & Rivlin (1962, 1964). We denote by I_λ $(\lambda = 1, 2, \ldots, \Lambda)$ the invariants of the integrity basis for the vector $\mathbf{U}$ and the system of vectors and tensors C, A, B, R; P_μ $(\mu = 1, 2, \ldots, N)$ denote the corresponding invariants which are linear and homogeneous in p_i. The expression (4·9) for $\mathscr{H}$ may then be re-written as

$$\mathscr{H} = \sum_{\mu=1}^{N} P_\mu \psi_\mu, \tag{4·10}$$

where ψ_μ are polynomials in I_λ with coefficients which are continuous single-valued functions of ρ_1, ρ_2, and by making use of (3·5) we obtain

$$\Psi_i = \sum_{\mu=1}^{N} \frac{\partial P_\mu}{\partial p_i} \psi_\mu. \tag{4·11}$$

If $\mathscr{S}_1$ is isotropic without a centre of symmetry (hemihedral) the invariants I_λ, P_μ are those appropriate to the proper orthogonal group of transformations defined by (4·4), (4·5). If $\mathscr{S}_1$ and $\mathscr{S}_2$ have a centre of symmetry at each point (holohedral) $\mathscr{H}$ is form-invariant also under the central inversion $\bar{X}_i = -X_i$, $\bar{y}_i = -y_i$. The invariants I_λ, P_μ are then those of the full orthogonal group specified by (4·4) (i) and (4·5) with $\det M = \pm 1$.

In the absence of interactions the stress components (3·1) become

$$\left.\begin{aligned}\boldsymbol{\sigma}_1 &= \alpha_1 I + \alpha_2 C + \alpha_3 C^2,\\ \boldsymbol{\sigma}_2 &= \beta_1 I + \beta_2 B + \beta_3 B^2,\end{aligned}\right\} \tag{4·12}$$

where α_i are polynomials in $\operatorname{tr} C$, $\operatorname{tr} C^2$ which depend also upon ρ_1 and β_i are polynomials in $\operatorname{tr} B$, $\operatorname{tr} B^2$, $\operatorname{tr} B^3$ (or $\det B$) which depend also upon ρ_2. Here we have assumed $\operatorname{tr} C^3$ (or $\det C$) to be eliminated from α_i by means of the relation

$$\rho_0/\rho_1 = [\det C]^{\frac{1}{2}}. \tag{4·13}$$

Forms similar to (4·12) may also be derived from (3·6), the only difference being that the coefficients α_i, β_i are then functions of both densities ρ_1, ρ_2.

* The functional form (4·9) is not, of course, the only one possible for $\mathscr{H}$. It follows from the earlier work that the nine components of the tensors A and R or of B and R may be replaced by those of the unsymmetrical tensor W with components

$$W_{ik} = \frac{\partial v_i}{\partial y_k} - \frac{\partial u_i}{\partial y_k};$$

$\mathscr{H}$ is again an isotropic function of its arguments.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

5. Diffusion through an isotropic rigid solid

If the solid $\mathscr{S}_1$ remains undeformed throughout the diffusion process we may take

$$y_i = M_{ik} X_k + \phi_i(t), \tag{5·1}$$

where $M_{ik} = M_{ik}(t)$ and $\phi_i(t)$ specify a rigid body rotation and translation respectively; M_{ik} are thus the elements of an orthogonal matrix satisfying (4·4). From (5·1) and (2·1) we have

$$\begin{aligned} u_i &= \frac{\partial M_{ik}}{\partial t} X_k + \frac{\partial \phi_i}{\partial t} \\ &= \frac{\partial M_{ik}}{\partial t} M_{lk}(y_l - \phi_l) + \frac{\partial \phi_i}{\partial t}, \end{aligned} \tag{5·2}$$

and hence, remembering (4·4),

$$\frac{\partial u_i}{\partial y_j} = \frac{\partial M_{ik}}{\partial t} M_{jk} = R^{(1)}_{ij}, \quad A_{ij} = 0. \tag{5·3}$$

Also from (5·1), (4·2) and (4·4)

$$\partial y_i / \partial X_k = M_{ik}, \quad C = I. \tag{5·4}$$

From these relations we see that for an isotropic rigid solid $\mathscr{S}_1$ the function (4·9) may be written

$$\mathscr{H} = \mathscr{H}(\mathbf{p};\, \rho_1, \rho_2;\, B, R, \mathbf{U}). \tag{5·5}$$

In this expression, $\mathbf{U}$ and R are calculated by using the forms (5·2) and (5·3) for $\mathbf{u}$ and R_1, respectively, and $\mathscr{H}$ is an isotropic function of the arguments indicated. The formulae (4·10) and (4·11) continue to apply but invariants involving C and A are now absent from the systems typified by I_λ, P_μ.

If $\mathscr{S}_1$ is at rest we have

$$\left.\begin{aligned} M = I, \quad \phi_i \equiv 0, \quad \mathbf{U} = \mathbf{v}, \\ R_1 = 0, \quad R = R_2, \end{aligned}\right\} \tag{5·6}$$

and (5·5) reduces to

$$\mathscr{H} = \mathscr{H}(\mathbf{p};\, \rho_1, \rho_2;\, B, R_2, \mathbf{v}). \tag{5·7}$$

This result for the diffusive force may be compared with that derived by a different method in the earlier work (Adkins 1963 *a*).

6. Steady-state diffusion through isotropic solid

The formula (4·11) for the diffusive force may be written out explicitly by using results derived by Spencer & Rivlin (1962, 1964) for isotropic integrity bases; the stresses may be treated similarly when interaction terms occur. The particular case of steady-state diffusion is of some importance and is considered here. Owing to the interaction, due to the form of the diffusive force, between the equations governing the deformation of $\mathscr{S}_1$ and those determining the flow of $\mathscr{S}_2$ we may expect that, in general, a time-dependent deformation would lead to a non-steady flow and that the converse would also apply. We therefore assume $\mathscr{S}_1$ to be in a state of static deformation and the equations of §§ 3 and 4 are simplified by the relations

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{U} = \mathbf{v}, \quad A = R_1 = 0, \quad R = R_2. \tag{6·1}$$

If we use the symbols $\mathbf{U}$ and R rather than $\mathbf{v}$ and R_2 the results apply to time-dependent deformations and non-steady flows provided that terms arising from non-zero values of A can be neglected.

For simplicity, we confine attention to the case where the diffusive force is a linear function of the components of $\mathbf{U}$ or $\mathbf{v}$ and their derivatives; for the same reason we assume terms arising from the rate of strain components B_{ik} to be absent from the expression for $\mathbf{\Psi}$*. The results are appropriate to the practically important case where the diffusion velocity is small, and the expressions for $\mathbf{\Psi}$ are then analogous to those given by Green & Adkins (1960) for the heat flux vector in thermoelasticity. If we define the components V_i of an axial vector $\mathbf{V}$ by the relations

$$V_i = \tfrac{1}{2}e_{kli}R_{kl}, \quad R_{ij} = e_{ijk}V_k, \tag{6·2}$$

where e_{ijk} are the components of the alternating tensor, the diffusive force, for a hemihedral material $\mathscr{S}_1$ is given by

$$\Psi_i = (\psi_1\delta_{ik}+\psi_2 C_{ik}+\psi_3 C_{il}C_{lk})\,U_k+(\psi_4\delta_{ik}+\psi_5 C_{ik}+\psi_6 C_{il}C_{lk})\,V_k, \tag{6·3}$$

where $\psi_1, \psi_2, \ldots, \psi_6$ are polynomials in $\operatorname{tr} C$ and $\operatorname{tr} C^2$ with coefficients which are continuous single-valued functions of ρ_1, ρ_2; we assume that $\operatorname{tr} C^3$ can be expressed in terms of $\operatorname{tr} C$, $\operatorname{tr} C^2$ and ρ_1 by using (4·13), without introducing singularities. In the case of holohedral materials $\mathscr{S}_1$, $\mathscr{S}_2$, (6·3) reduces to

$$\Psi_i = (\psi_1\delta_{ik}+\psi_2 C_{ik}+\psi_3 C_{il}C_{lk})\,U_k. \tag{6·4}$$

7. Swelling of a solid by absorption of fluid

The preceding theory may evidently be employed to examine static deformations of a solid into which a fluid has been absorbed. This kind of problem arises in the swelling of rubber and plastics by solvents and of fibres and wood by water and other fluids. In this static case

$$\mathbf{u} = \mathbf{v} = \mathbf{U} = \mathbf{0}, \quad A = B = R = 0, \quad \mathbf{\Psi} = \mathbf{0}, \tag{7·1}$$

and the stresses (4·12) become

$$\boldsymbol{\sigma} = \alpha_1 I+\alpha_2 C+\alpha_3 C^2, \quad \boldsymbol{\sigma}_2 = \beta_1 I. \tag{7·2}$$

The equations of motion for $\mathscr{S}_2$ yield

$$\partial\beta_1/\partial y_i = 0 \quad \text{or} \quad \beta_1 = \text{constant} = k \text{ (say)}, \tag{7·3}$$

and with these we must associate (4·13).

If no interactions are assumed in the expressions for the stresses, the coefficients α_i may be regarded as functions of ρ_1, $\operatorname{tr} C$ and $\operatorname{tr} C^2$ while β_1 becomes a function only of ρ_2. Equation (7·3) would then imply that the fluid is uniformly distributed throughout $\mathscr{S}_1$ even when the deformation of $\mathscr{S}_1$ is inhomogeneous. This difficulty does not arise when interaction terms are included. For example, if we assume forms for the stresses based on (3·6) in which α_i and β_i depend upon both ρ_1 and ρ_2, equation (7·3) yields a relation between the densities which may, in principle, be solved for ρ_2 in terms of ρ_1. In view of (4·13), the equations of equilibrium $\partial\sigma^{(1)}_{ik}/\partial y_k = 0$ furnish, in general, three equations to determine the deformation of $\mathscr{S}_1$. This is the kind of problem which arises in the theory of finite elasticity. When this has been solved, ρ_1 and ρ_2 are given as functions of position by (4·13) and (7·3). A similar procedure may be employed when more general interaction terms occur in the formulae for $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$.

* These would give rise to additional terms in (6·3), (6·4) remaining unchanged.

In the case of pure homogeneous deformation with principal directions parallel to the X_i axes we have

$$\left.\begin{aligned}(y_1, y_2, y_3) &= (\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3),\\ (C_{11}, C_{22}, C_{33}) &= (\lambda_1^2, \lambda_2^2, \lambda_3^2), \quad C_{ij} = 0 \quad (i \neq j),\end{aligned}\right\} \tag{7·4}$$

where $\lambda_1, \lambda_2, \lambda_3$ are constants. Equations (7·2) and (4·13) yield

$$\sigma_{ii}^{(1)} = \alpha_1 + \alpha_2 \lambda_i^2 + \alpha_3 \lambda_i^4 \quad (i \text{ not summed}), \tag{7·5}$$

$$\lambda_1 \lambda_2 \lambda_3 \rho_1 = \rho_0. \tag{7·6}$$

If the swollen body is a cube, with plane faces normal to the X_i directions subject to uniform pressures p_i we have

$$\sigma_{ii}^{(1)} + \sigma_{ii}^{(2)} = \alpha_1 + \alpha_2 \lambda_i^2 + \alpha_3 \lambda_i^4 + \beta_1 = -p_i \quad (i \text{ not summed}). \tag{7·7}$$

If the density ρ_2 and the pressures p_i are prescribed, the relations (7·6) and (7·7) may be regarded as four equations for the determination of λ_i and ρ_1.

When the solid-fluid mixture is at rest under the action of no extraneous forces we may take

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda \text{ (say)}, \quad p_i = 0, \tag{7·8}$$

and equations (7·6), (7·7) yield

$$\lambda^3 \rho_1 = \rho_0, \quad \alpha_1 + \alpha_2 \lambda^2 + \alpha_3 \lambda^4 + \beta_1 = 0. \tag{7·9}$$

In this case we have two relations connecting λ, ρ_1 and ρ_2.

For simple extension in the X_1 direction we have

$$\lambda_2 = \lambda_3, \quad -p_1 = T \text{ (say)}, \quad p_2 = p_3 = 0, \tag{7·10}$$

and then from (7·6) and (7·7)

$$\left.\begin{aligned}\lambda_1 \lambda_2^2 \rho_1 &= \rho_0,\\ \alpha_1 + \alpha_2 \lambda_1^2 + \alpha_3 \lambda_1^4 + \beta_1 &= T,\\ \alpha_1 + \alpha_2 \lambda_2^2 + \alpha_3 \lambda_2^4 + \beta_1 &= 0.\end{aligned}\right\} \tag{7·11}$$

These may be regarded as three relations connecting λ_1, λ_2, ρ_1 and ρ_2.

The problem of swelling for rubber-like materials has been examined from a somewhat different point of view by Treloar (1950, 1958). From thermodynamic and molecular considerations he has derived an expression for the free energy of a swollen polymer in terms of the deformation and the volume fraction of solid polymer present in the solid-fluid mixture. From this there follows an expression for the total stress $\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2$. Treloar examines various kinds of homogeneous deformation using this approach and concentrates on the problem of equilibrium swelling in which the solid has absorbed the maximum amount of fluid consistent with its molecular structure.

8. Diffusion through uniformly stretched isotropic elastic plate

The theory of § 6 is here employed to examine the steady-state diffusion of a fluid normally through an isotropic elastic plate or slab under uniform all-round extension or compression. The plate is bounded in the undeformed state by the plane faces $X_1 = a_1$, $X_1 = a_2$ and the deformation is defined by

$$y_1 = \lambda(X_1), \quad y_2 = \mu X_2, \quad y_3 = \mu X_3, \tag{8·1}$$

where μ is a constant. The velocity components of the fluid at the point y_i are given by

$$(v_1, v_2, v_3) = (u, 0, 0), \tag{8·2}$$

where u is a function only of y_1 (or X_1).

In this and subsequent sections we restrict attention to the case where there are no interaction terms in the stresses which then take the forms (4·12). The results of § 7 suggest that this assumption may, in practice, need some modification, but even with the simplified theory some interesting results emerge from the analysis. An extension to the more general theory presents no difficulty of principle. From (4·2) and (8·1) we now have

$$C_{11} = \lambda'^2, \quad C_{22} = C_{33} = \mu^2, \quad C_{ij} = 0 \quad (i \neq j), \tag{8·3}$$

where, here and subsequently, a prime denotes differentiation with respect to X_1. The strain invariants are given by

$$\operatorname{tr} C = \lambda'^2 + 2\mu^2, \quad \operatorname{tr} C^2 = \lambda'^4 + 2\mu^4, \tag{8·4}$$

and since $\det C = \lambda'^2\mu^4$

$$\rho_1 \lambda' \mu^2 = \rho_0. \tag{8·5}$$

From (4·2) and (8·2)

$$B_{11} = u'/\lambda', \quad B_{22} = B_{33} = B_{12} = B_{23} = B_{31} = 0 \tag{8·6}$$

and hence

$$\operatorname{tr} B = u'/\lambda', \quad \operatorname{tr} B^2 = u'^2/\lambda'^2, \quad \operatorname{tr} B^3 = u'^3/\lambda'^3. \tag{8·7}$$

The stresses (4·12) take the forms

$$\left.\begin{aligned} \sigma^{(1)}_{11} &= \alpha_1 + \alpha_2 \lambda'^2 + \alpha_3 \lambda'^4, \\ \sigma^{(1)}_{22} &= \sigma^{(1)}_{33} = \alpha_1 + \alpha_2 \mu^2 + \alpha_3 \mu^4, \quad \sigma^{(1)}_{ij} = 0 \quad (i \neq j), \end{aligned}\right\} \tag{8·8}$$

$$\left.\begin{aligned} \sigma^{(2)}_{11} &= \beta_1 + \beta_2 u'/\lambda' + \beta_3 u'^2/\lambda'^2, \\ \sigma^{(2)}_{22} &= \sigma^{(2)}_{33} = \beta_1, \quad \sigma^{(2)}_{ij} = 0 \quad (i \neq j). \end{aligned}\right\} \tag{8·9}$$

In (8·8) the coefficients α_i are polynomials in the invariants (8·4) and depend also upon ρ_1. In view of (8·5) we may regard α_i as functions of λ' and μ and we shall assume that they are continuous, single-valued functions of these arguments within the range considered; the constant μ will not be exhibited explicitly in the subsequent analysis. The coefficients β_i are polynomials in the invariants (8·7) which depend also upon ρ_2; we shall assume that they are continuous single-valued functions of u'/λ' and ρ_2.

From (6·2) and (8·2), $\mathbf{V} = \mathbf{0}$, (or $R = 0$) and the diffusive force components assume identical forms for holohedral and hemihedral materials. From (8·2), (8·3) and (6·3) or (6·4) we have

$$\Psi_1 = \Upsilon u, \quad \Psi_2 = \Psi_3 = 0, \tag{8·10}$$

where

$$\Upsilon = \psi_1 + \psi_2 \lambda'^2 + \psi_3 \lambda'^4 \tag{8·11}$$

is a function of ρ_1, ρ_2 and λ'^2 that is, of ρ_2 and λ'.

The equations of motion (2·7) reduce to

$$\left.\begin{aligned} \mathrm{d}\sigma^{(1)}_{11}/\mathrm{d}y_1 - \rho_2 \Upsilon u &= 0, \\ \mathrm{d}\sigma^{(2)}_{11}/\mathrm{d}y_1 + \rho_2 \Upsilon u &= \rho_2 u (\mathrm{d}u/\mathrm{d}y_1), \end{aligned}\right\} \tag{8·12}$$

for the solid and fluid respectively, the remaining equations being satisfied identically. The equation of continuity (2·3) yields

$$\mathrm{d}(\rho_2 u)/\mathrm{d}y_1 = 0 \quad \text{or} \quad \rho_2 u = k, \tag{8·13}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

where k is a constant. By adding (8·12) and making use of (8·13) we obtain

$$\sigma_{11}^{(1)} + \sigma_{11}^{(2)} = ku + k_1, \tag{8·14}$$

k_1 being a further constant of integration. The first of (8·12) and (8·14) may be written

$$\left.\begin{aligned} \mathrm{d}\sigma_{11}^{(1)}/\mathrm{d}X_1 - k\Upsilon\lambda' &= 0, \\ \sigma_{11}^{(1)} + \sigma_{11}^{(2)} &= k^2/\rho_2 + k_1, \end{aligned}\right\} \tag{8·15}$$

and in view of (8·8), (8·9) and (8·13) these may be regarded as two differential equations for ρ_2 and λ. When these have been solved, u and ρ_1 are given by (8·13) and (8·5), respectively.

If, within the solid $\mathscr{S}_1$, the fluid $\mathscr{S}_2$ behaves as an ideal fluid so that the stress $\boldsymbol{\sigma}_2$ takes the form

$$\boldsymbol{\sigma}_2 = F(\rho_2)\, I, \tag{8·16}$$

where F is a known function of ρ_2*, then from the second of (8·15)

$$\sigma_{11}^{(1)} = k^2/\rho_2 - F(\rho_2) + k_1. \tag{8·17}$$

In the corresponding problem of diffusion through a rigid plate, the classical result based upon Fick's law is derived from the more general theory by making the assumptions that k is small, that F is linear in ρ_2 and that Υ is constant (Adkins 1963 *a*). In the present instance, if Υ is constant, the first of (8·15) integrates to give

$$\sigma_{11}^{(1)} = k\Upsilon\lambda + \text{constant}, \tag{8·18}$$

and this may be regarded as a first-order differential equation for λ which does not involve X_1 explicitly. An expression for λ in terms of ρ_2 is obtained from (8·17) and (8·18).

Returning to the general equations (8·8) to (8·15), it would appear natural to examine the possibility of a solution in which the deformation and velocity field are almost uniform throughout the plate. However, if we write

$$\left.\begin{aligned} &y_1 = \lambda(X_1) = \lambda_0 X_1 + \epsilon\bar{\lambda}, \quad u = u_0 + \epsilon\bar{u}, \\ &\rho_1 = c_1 + \epsilon\bar{\rho}_1, \quad \rho_2 = c_2 + \epsilon\bar{\rho}_2 \quad (\lambda_0, u_0, c_1, c_2 \text{ constant}), \end{aligned}\right\} \tag{8·19}$$

where ϵ ($\ll 1$) is a small real parameter which is independent of ρ_1, ρ_2, u and λ and expand (8·8) to (8·15) in powers of ϵ, the terms independent of ϵ in the first of (8·15) can vanish only if $\Upsilon(c_2, \lambda_0) = 0$. Since this implies a relationship between c_2 and λ_0 which it may not be possible to satisfy for relevant values of these quantities, we shall not consider this case further.

This difficulty does not arise if the velocity u is small and of order ϵ. The assumption of small velocity is made in obtaining the classical solution for diffusion through a rigid plate and to a first approximation the density of the diffusing substance is a linear function of position; a further approximation based upon this classical solution has been derived by Adkins (1963 *a*). Guided by the solution for a rigid plate, in place of (8·19) we write

$$\left.\begin{aligned} u = \epsilon u_0 + \epsilon^2\bar{u} + O(\epsilon^3), \quad \lambda = \lambda_0 X_1 + \epsilon\bar{\lambda} + O(\epsilon^2), \\ \rho_1 = c_1 + \epsilon\bar{\rho}_1 + O(\epsilon^2), \quad \rho_2 = c_2 + \epsilon\bar{\rho}_2 + O(\epsilon^2), \end{aligned}\right\} \tag{8·20}$$

where u_0, λ_0, c_1, c_2 are again constants and $\bar{u}$, $\bar{\lambda}$, $\bar{\rho}_1$, $\bar{\rho}_2$ are functions of X_1.

* In the more general case, when interactions occur F may be a function also of the invariants (8·4)

Introducing (8·20) into (8·5) and (8·13) and equating separately to zero coefficients of corresponding powers of ϵ we obtain

$$\lambda_0\mu^2 c_1 = \rho_0, \quad c_2 u_0 = k_0, \tag{8·21}$$

$$\lambda_0\bar{\rho}_1 + c_1\bar{\lambda}' = 0, \quad \bar{\rho}_2 u_0 + c_2\bar{u} = \bar{k}, \tag{8·22}$$

k_0 and $\bar{k}$ being constants such that $\epsilon k_0 + \epsilon^2\bar{k} = k$.

If we replace k_1 by $k_1 + \epsilon\bar{k}_1$ in the second of (8·15) and assume that the functions α_i, β_i, Υ can be expanded by Taylor's theorem in ascending powers of ϵ the terms independent of ϵ yield

$$[\alpha_1 + \beta_1 + \lambda_0^2\alpha_2 + \lambda_0^4\alpha_3]_0 = k_1. \tag{8·23}$$

Here and subsequently the symbol $[\ ]_0$ indicates that the quantities inside the square brackets are evaluated at $\epsilon = 0$. Terms independent of ϵ in the first of (8·15) vanish identically.

Similarly, from the terms linear in ϵ in the expansions of (8·15) we obtain

$$\left.\begin{aligned} a\bar{\lambda}'' &= \kappa_1, \\ a\bar{\lambda}' + b\bar{\rho}_2 &= \bar{k}_1, \end{aligned}\right\} \tag{8·24}$$

where

$$\left.\begin{aligned} a &= \left[\frac{\partial\alpha_1}{\partial\lambda'} + \lambda_0^2\frac{\partial\alpha_2}{\partial\lambda'} + \lambda_0^4\frac{\partial\alpha_3}{\partial\lambda'} + 2\lambda_0\alpha_2 + 4\lambda_0^3\alpha_3\right]_0, \\ b &= \left[\frac{\partial\beta}{\partial\rho_2}\right]_0, \quad \kappa_1 = k_0\lambda_0[\Upsilon]_0 \end{aligned}\right\} \tag{8·25}$$

are constants. From (8·24) we obtain

$$\left.\begin{aligned} a\bar{\lambda} &= \tfrac{1}{2}\kappa_1 X_1^2 + \kappa_2 X_1 + \kappa_3, \\ b\bar{\rho}_2 &= \bar{k}_1 - \kappa_2 - \kappa_1 X_1. \end{aligned}\right\} \tag{8·26}$$

As in the classical theory for a rigid plate, the density ρ_2 is a linear function of position to this order of approximation. Higher-order approximations may be derived by considering the coefficients of higher powers of ϵ in (8·20) and the subsequent equations. In this case, however, terms non-linear in u if they exist, in the constitutive equations for $\boldsymbol{\sigma}_2$ and $\boldsymbol{\Psi}$ would need to be taken into account. The constants of integration are determined from the boundary conditions at $X_1 = a_1$, $X_1 = a_2$. In view of (8·20), there is no loss of generality in taking $\bar{k}_1$, κ_2 and κ_3 to be zero in (8·26).

9. Steady flow through a sheared slab

Steady-state diffusion through a slab or plate of isotropic elastic material in shear may be examined similarly. We again suppose the plate to be bounded by the planes $X_1 = a_1$, $X_1 = a_2$ and consider a steady-state deformation in which planes X_1 = constant are sheared relative to each other in directions in the X_2, X_3 planes, together with a displacement along their normal. The deformation may therefore be defined by

$$y_1 = H(X_1), \quad y_2 = X_2 + K(X_1), \quad y_3 = X_3 + L(X_1), \tag{9·1}$$

where H, K, L are functions of X_1. We assume that during diffusion through the plate, the fluid receives components of velocity in the X_2, X_3 directions as a result of the deformation of the solid. The velocity components v_i at the point y_i are then

$$(v_1, v_2, v_3) = (u, v, w), \tag{9·2}$$

where u, v, w are functions only of y_1 (or X_1).

Again denoting differentiation with respect to X_1 by a prime, we have from (4·2) and (9·1),

$$C_{ik} = \begin{bmatrix} H'^2 & H'K' & H'L' \\ H'K' & 1+K'^2 & K'L' \\ H'L' & K'L' & 1+L'^2 \end{bmatrix} \tag{9·3}$$

$$\operatorname{tr} C = 2+q, \quad \operatorname{tr} C^2 = (q+1)^2+1-2H'^2, \quad \det C = H'^2, \tag{9·4}$$

with
$$q = H'^2+K'^2+L'^2. \tag{9·5}$$

Also, from (9·1), (9·2) and (4·2)

$$H'B_{ij} = \begin{bmatrix} u' & \tfrac{1}{2}v' & \tfrac{1}{2}w' \\ \tfrac{1}{2}v' & 0 & 0 \\ \tfrac{1}{2}w' & 0 & 0 \end{bmatrix}, \quad H'R_{ij} = \begin{bmatrix} 0 & -\tfrac{1}{2}v' & -\tfrac{1}{2}w' \\ \tfrac{1}{2}v' & 0 & 0 \\ \tfrac{1}{2}w' & 0 & 0 \end{bmatrix}, \tag{9·6}$$

$$\operatorname{tr} B = u'/H', \quad \operatorname{tr} B^2 = [2u'^2+v'^2+w'^2]/(2H'^2), \quad \det B = 0, \tag{9·7}$$

and from (9·4) and (4·13)
$$\rho_1 H' = \rho_0. \tag{9·8}$$

From (4·12) and (9·3) the stresses $\sigma_{ij}^{(1)}$ become

$$\left.\begin{aligned} \sigma_{11}^{(1)} &= \alpha_1+(\alpha_2+\alpha_3 q)\,H'^2, \\ \sigma_{22}^{(1)} &= \alpha_1+\alpha_2(1+K'^2)+\alpha_3[1+(2+q)\,K'^2], \\ \sigma_{12}^{(1)} &= H'K'\chi, \quad \sigma_{13}^{(1)} = H'L'\chi, \quad \sigma_{23}^{(1)} = K'L'(\chi+\alpha_3), \end{aligned}\right\} \tag{9·9}$$

where
$$\chi = \alpha_2+(1+q)\,\alpha_3, \tag{9·10}$$

and the formula for $\sigma_{33}^{(1)}$ is derived from that for $\sigma_{22}^{(1)}$ by replacing K' by L' throughout. The coefficients α_i in (9·9) and (9·10) are functions of the invariants (9·4) and ρ_1. Alternatively, remembering (9·8), they may be regarded as functions of H' and $K'^2+L'^2$.

Again, from (4·12) and (9·6) we obtain

$$\left.\begin{aligned} &\sigma_{11}^{(2)} = \beta_1+\beta_2 u'/H'+\beta_3(4u'^2+v'^2+w'^2)/(4H'^2), \\ &\sigma_{22}^{(2)} = \beta_1+\beta_3 v'^2/(4H'^2), \quad \sigma_{33}^{(2)} = \beta_1+\beta_3 w'^2/(4H'^2), \\ &\sigma_{12}^{(2)} = v'(\beta_2 H'+\beta_3 u')/(2H'^2), \quad \sigma_{13}^{(2)} = w'(\beta_2 H'+\beta_3 u')/(2H'^2), \\ &\sigma_{23}^{(2)} = v'w'\beta_3/(4H'^2). \end{aligned}\right\}. \tag{9·11}$$

The coefficients β_i are functions of ρ_2 and the invariants (9·7), that is, of ρ_2, u'/H' and $(v'^2+w'^2)/H'^2$.

The diffusive force components for a hemihedral isotropic material $\mathscr{S}_1$ are given by (6·3). With the help of (9·3), (6·2) and (9·6) we obtain

$$\left.\begin{aligned} \Psi_1 &= \psi_{11}u+Q_1H'(K'v+L'w)-\overline{Q}_1(L'v'-K'w')/2, \\ \Psi_2 &= \psi_{22}v+K'(Q_1H'u+Q_2L'w)-(\overline{Q}_2K'L'v'-\overline{\psi}_{22}w')/(2H'), \\ \Psi_3 &= \psi_{33}w+L'(Q_1H'u+Q_2K'v)-(\overline{\psi}_{33}v'-K'L'\overline{Q}_2w')/(2H'), \end{aligned}\right\} \tag{9·12}$$

where
$$\left.\begin{aligned} \psi_{11} &= \psi_1+H'^2\psi_2+qH'^2\psi_3, \\ \psi_{22} &= \psi_1+(1+K'^2)\,\psi_2+[1+(2+q)\,K'^2]\,\psi_3, \\ Q_1 &= \psi_2+(1+q)\,\psi_3, \quad Q_2 = Q_1+\psi_3, \end{aligned}\right\} \tag{9·13}$$

ψ_{33} is derived from ψ_{22} by replacing K' by L' throughout and the barred quantities $\overline{\psi}_{22}$, $\overline{\psi}_{33}$, $\overline{Q}_1$, $\overline{Q}_2$ are derived from the corresponding unbarred quantities by replacing ψ_1, ψ_2, ψ_3 by ψ_4, ψ_5, ψ_6, respectively.

Remembering (9·1) and (9·2), it is seen without difficulty that the stresses and diffusive force components are all functions of X_1 or y_1. The equations of motion (2·7) therefore yield

$$\mathrm{d}\sigma_{i1}^{(1)}/\mathrm{d}y_1-\rho_2\Psi_i = 0 \quad (i = 1, 2, 3), \tag{9·14}$$

$$\mathrm{d}\sigma_{i1}^{(2)}/\mathrm{d}y_1+\rho_2\Psi_i = \rho_2 u(\mathrm{d}v_i/\mathrm{d}y_1) \quad (i = 1, 2, 3), \tag{9·15}$$

v_i being given by (9·2).

The equation of continuity (2·3) again gives

$$\mathrm{d}(\rho_2 u)/\mathrm{d}y_1 = 0 \quad \text{or} \quad \rho_2 u = k \; (k = \text{constant}), \tag{9·16}$$

and from (9·14) to (9·16) we obtain

$$\sigma_{i1}^{(1)}+\sigma_{i1}^{(2)} = kv_i+k_i, \tag{9·17}$$

k_i being further constants of integration.

If the expressions (9·9), (9·11) and (9·12) for $\sigma_{i1}^{(1)}$, $\sigma_{i1}^{(2)}$ and Ψ_i are introduced into (9·14) and (9·17) and ρ_1 and ρ_2 are eliminated by means of (9·8) and (9·16), then these relations may be regarded as six ordinary differential equations for the determination of the quantities H, K, L, u, v, w describing the deformation and flow.

If the deformation and flow are assumed to take place in the X_1, X_2 plane, so that $L(X_1) \equiv 0$, $w(X_1) \equiv 0$ then from (9·9), (9·11) and (9·12)

$$\sigma_{13}^{(1)} = \sigma_{13}^{(2)} = 0, \quad \Psi_3 = -v'\bar{\psi}_{33}/(2H').$$

Provided $v' \neq 0$, the third equation of (9·14) and of (9·15) can then be satisfied if and only if $\bar{\psi}_{33} = 0$; this occurs if the solid $\mathscr{S}_1$ has a centre of symmetry. Otherwise, if (9·14) and (9·15) are not to form an overdetermined system, neither L' nor w can vanish. We conclude that, in general, a fluid diffusing through a hemihedral elastic solid in shear is deflected in a direction normal to the plane of shear; this effect is absent in holohedral materials.

We may observe that even if $\mathscr{S}_1$ has a centre of symmetry at each point the system of equations (9·14), (9·15) becomes overdetermined if we assume only one of the variables L or w to be zero. In general, therefore, a shearing deformation is accompanied by a flow of fluid in the direction of shear, and conversely, a flow of fluid in a direction tangential to the X_2, X_3 planes is accompanied by a shearing deformation in that direction.

10. Slow steady flow through a sheared slab

When the diffusion velocities are sufficiently small, and the deformation of $\mathscr{S}_1$ differs only slightly from simple shear, an explicit solution of the equations of § 9 may be obtained by the method of § 8. We assume the shear of $\mathscr{S}_1$, in the absence of $\mathscr{S}_2$ to be a simple shear in the X_1, X_2 plane and guided by the results of § 8 we replace (9·1) and (9·2) by

$$\left.\begin{aligned} &y_1 = X_1+\epsilon\bar{H}(X_1), \quad y_2 = X_2+K_0X_1+\epsilon\bar{K}(X_1), \quad y_3 = X_3+\epsilon\bar{L}(X_1), \\ &v_1 = \epsilon u_0+\epsilon^2\bar{u}, \quad v_2 = \epsilon v_0+\epsilon^2\bar{v}, \quad v_3 = \epsilon w_0+\epsilon^2\bar{w}, \end{aligned}\right\} \tag{10·1}$$

In (10·1) ϵ is a small parameter independent of X_i, K_0 is a constant, $\bar{H}$, $\bar{K}$, $\bar{L}$ and the velocities are functions of X_1 and terms of higher order in ϵ have been neglected. As in § 8, we assume the densities ρ_1, ρ_2 to be given by

$$\rho_1 = c_1+\epsilon\bar{\rho}_1, \quad \rho_2 = c_2+\epsilon\bar{\rho}_2, \tag{10·2}$$

where c_1 and c_2 are constants, and replace k by $\epsilon k_0+\epsilon^2\bar{k}$ in (9·16).

Equations (9·8) and (9·16) then yield

$$c_1 = \rho_0, \quad c_2 u_0 = k_0, \tag{10·3}$$

$$\bar{\rho}_1 + c_1 H' = 0, \quad \bar{\rho}_2 u_0 + c_2 \bar{u} = \bar{k}. \tag{10·4}$$

In (9·9) and (9·10) we regard α_i as functions of H', K' and L'; in (9·11) we regard β_i as functions of ρ_2, u'/H' and $(v'^2+w'^2)/H'^2$. Assuming Taylor series expansions in powers of ϵ to be valid, we then have

$$\left.\begin{aligned} \alpha_i &= [\alpha_i]_0 + \epsilon\left\{\bar{H}'\left[\frac{\partial \alpha_i}{\partial H'}\right]_0 + \bar{K}'\left[\frac{\partial \alpha_i}{\partial K'}\right]_0 + \bar{L}'\left[\frac{\partial \alpha_i}{\partial L'}\right]_0\right\} + O(\epsilon^2), \\ \beta_i &= [\beta_i]_0 + \epsilon\bar{\rho}_2\left[\frac{\partial \beta_i}{\partial \rho_2}\right]_0 + O(\epsilon^2), \end{aligned}\right\} \tag{10·5}$$

the symbol $[\ ]_0$ again indicating that the quantities inside brackets are evaluated at $\epsilon = 0$.

The terms independent of ϵ in (9·14) vanish identically, and those independent of ϵ in (9·17) merely yield relations connecting $[\alpha_i]_0$ and $[\beta_i]_0$ with arbitrary constants. From the terms linear in ϵ we obtain

$$\left.\begin{aligned} & a_1 \bar{H}'' + a_2 \bar{K}'' + a_3 \bar{L}'' + d_1 v_0 + d_2 w_0' = d_3, \\ & a_4 \bar{H}'' + a_5 \bar{K}'' + a_6 \bar{L}'' + d_4 v_0 + d_5 w_0' = d_6, \\ & a_7 \bar{L}'' + d_7 v_0' + d_8 w_0 = 0, \\ & a_1 \bar{H}' + a_2 \bar{K}' + a_3 \bar{L}' + b_1 \bar{\rho}_2 = \bar{k}_1, \\ & a_4 \bar{H}' + a_5 \bar{K}' + a_6 \bar{L}' + b_2 v_0' = \bar{k}_2, \\ & a_7 \bar{L}' + b_2 w_0' = \bar{k}_3, \end{aligned}\right\} \tag{10·6}$$

where

$$\left.\begin{aligned} & (a_1, a_2, a_3) = \left(\frac{\partial \mathscr{A}}{\partial H'}, \frac{\partial \mathscr{A}}{\partial K'}, \frac{\partial \mathscr{A}}{\partial L'}\right), \quad \mathscr{A} = \alpha_1 + (\alpha_2 + \alpha_3 q) H'^2, \\ & a_4 = K_0\left\{a_7 + \left[\frac{\partial \chi}{\partial H'}\right]_0\right\}, \quad a_5 = a_7 + K_0\left[\frac{\partial \chi}{\partial K'}\right]_0, \quad a_6 = K_0\left[\frac{\partial \chi}{\partial L'}\right]_0, \quad a_7 = [\chi]_0, \\ & b_1 = [\partial \beta_1/\partial \rho_2]_0, \quad b_2 = [\beta_2]_0/2, \end{aligned}\right\} \tag{10·7}$$

$$\left.\begin{aligned} & d_1 = -c_2 K_0 [Q_1]_0, \quad 2d_2 = -c_2 K_0 [\bar{Q}_1]_0, \quad d_3 = c_2 u_0 [\psi_{11}]_0, \\ & d_4 = -c_2 [\psi_{22}]_0, \quad 2d_5 = -c_2 [\bar{\psi}_{22}]_0, \quad d_6 = -d_1 u_0, \\ & 2d_7 = c_2 [\bar{\psi}_{33}]_0, \quad d_8 = -c_2 [\psi_{33}]_0, \end{aligned}\right\} \tag{10·8}$$

and the constants $\bar{k}_i$ may be chosen arbitrarily owing to the arbitrary nature of the constants k_i in (9·17).

Solutions of (10·6) may readily be obtained in the form

$$\left.\begin{aligned} \bar{H} &= \sum_{r=1}^{4} \eta_r e^{m_r X_1} + \eta_5 X_1^2 + \eta_6 X_1 + \eta_7, \\ \bar{K} &= \sum_{r=1}^{4} \kappa_r e^{m_r X_1} + \kappa_5 X_1^2 + \kappa_6 X_1 + \kappa_7, \\ \bar{L} &= \sum_{r=1}^{4} \lambda_r e^{m_r X_1} + \lambda_5 X_1 + \lambda_6, \\ v_0 &= \sum_{r=1}^{4} \xi_r e^{m_r X_1} + d_6/d_4, \\ w_0 &= \sum_{r=1}^{4} \zeta_r e^{m_r X_1}, \end{aligned}\right\} \tag{10·9}$$

where the constants η_r, κ_r, λ_r, ξ_r, ζ_r satisfy the equations

$$\left.\begin{aligned} &a_1 m_r^2 \eta_r + a_2 m_r^2 \kappa_r + a_3 m_r^2 \lambda_r + d_1 \xi_r + d_2 m_r \zeta_r = 0,\\ &a_4 m_r^2 \eta_r + a_5 m_r^2 \kappa_r + a_6 m_r^2 \lambda_r + d_4 \xi_r + d_5 m_r \zeta_r = 0,\\ &a_7 m_r^2 \lambda_r + d_7 m_r \xi_r + d_8 \zeta_r = 0,\\ &a_4 \eta_r + a_5 \kappa_r + a_6 \lambda_r + b_2 \xi_r = 0,\\ &a_7 \lambda_r + b_2 \zeta_r = 0,\\ &\qquad (r = 1 \text{ to } 4;\ r \text{ not summed}),\end{aligned}\right\} \tag{10·10}$$

and for these to be compatible, m_r are the roots of

$$b_2^2 m^4 - \{b_2(d_4+d_2) + d_5 d_7\} m^2 + d_4 d_8 = 0, \tag{10·11}$$

provided $a_1 a_5 - a_2 a_4 \neq 0$ and $a_7 \neq 0$. The constants η_5, κ_5, λ_5 in (10·9) are given by

$$2\eta_5 = \frac{a_5(d_3 d_4 - d_1 d_6)}{d_4(a_1 a_5 - a_2 a_4)}, \quad \kappa_5 = -\frac{a_4 \eta_5}{a_5}, \quad \lambda_5 = \frac{\bar{k}_3}{a_7},$$

and

$$a_4 \eta_6 + a_5 \kappa_6 = \bar{k}_2 - a_6 \bar{k}_3 / a_7;$$

η_7, κ_7 and λ_6, which specify a rigid body displacement may be chosen arbitrarily and since $\bar{k}_2$, $\bar{k}_3$ are arbitrary constants, η_6 and κ_6 are also arbitrary. The arbitrary constants in (10·9) are chosen to satisfy the boundary conditions at $X_1 = a_1$, $X_1 = a_2$. The density $\bar{\rho}_2$ is given by the fourth equation of (10·6) and $\bar{u}$, $\bar{\rho}_1$ are determined from (10·4).

If the constants η_r, κ_r, λ_r, ξ_r, ζ_r ($r = 1$ to 4) are chosen to be zero in (10·9) the solution for ρ_2 and the normal flow velocity u resembles that for normal diffusion through an isotropic rigid plate (Adkins 1963*a*). In addition, however, a flow is induced parallel to the direction of the finite shear, and a small shearing deformation occurs normal to the plane of the finite shear. We observe that the second-order velocities $\bar{v}$, $\bar{w}$ must be determined from the next stage of the approximation process, and terms non-linear in v_i, if they occur in the constitutive equations for the stress and diffusive force, would then need to be taken into account.

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